
Safety and Reliability of Embedded Systems

(Sicherheit und Zuverlässigkeit eingebetteter Systeme)

Fault Tree Analysis

Mathematical Background and Algorithms

Content

- ☐ Definitions of Terms
- ☐ Introduction to Combinatorics
- ☐ General Formulas for AND-, OR-, NOT-, XOR-Gates
- ☐ Calculation of Top-Event Probability
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Definitions of Terms

- ☐ **Failure** is any behavior of a component or system that deviates from the specification
- ☐ **Fault** is an abnormal state or condition within a component that can lead to a failure
- ☐ **Accident** is an undesired event that causes death or injury of persons or harm to goods or to the environment
- ☐ **Hazard** is a state of a system *and* its environment where the occurrence of an accident depends only on influences that are not controllable by the system
- ☐ **Risk** is the combination of hazard probability and severity of the resulting accident
- ☐ **Acceptable Risk** is a level of risk that authorities or other bodies have defined as acceptable according to acceptance criteria



Definitions of Terms

**Other definitions exist,
but many of them are impractical**

☐ **Safety** is freedom from unacceptable risks

☞ **Safety analysis** aims at proving that the actual risk is below the acceptable risk

☐ **Availability** is the property of a system to fulfill its purpose at a given point in time / is the probability that the system fulfills its purpose at a given point in time

☞ **The focus is on uninterrupted service**

☐ **Reliability** is the property of an entity to fulfill its reliability requirements during or after a given time span under given application conditions

☞ **Reliability** is related to the probability of a failure event over the mission time

Modeling of Reliability

☐ Reliability Function $R(t)$:

- $F(t)$ gives the probability that at time t the (non-repairable) system has failed
- Thus $R(t) = 1 - F(t)$ is the probability that at time t no failure has occurred yet

☐ Probability Density $f(t)$:

- The probability density $f(t)$ describes the modification of the probability that a system fails over time:

$$f(t) = \frac{d F(t)}{dt}$$



States have a probability.
Events have a probability density
and an (occurrence) rate

☐ Failure Rate:

- The failure rate is the relative boundary value of failed entities at time t in a time interval that approximates zero, referring to the entities still functional at the beginning of the time interval:

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{dF(t) / dt}{R(t)} = \frac{-dR(t) / dt}{R(t)}$$

Cut Sets, Minimal Cut Sets, Path Sets

- ☐ A **Cut Set** is a set of basic events, which in conjunction cause the top event
- ☐ A **Minimal Cut Set (MCS)** is a cut set that no longer is a cut set if any of its basic events is removed
- ☐ A **Path Set** is a set of basic events that, if they are false, inhibit the top event from occurring
- ☐ A **Minimal Path Set (MPS)** is a path set that no longer is a path set if any of its basic events is removed

Introduction to Combinatorics: Truth Values

AND: $A \wedge B$

OR: $A \vee B$

Proposition $A \rightarrow$	False	True
Proposition $B \downarrow$		
False	False	False
	False	True
True	False	True
	True	True

👉 True and False are often represented by 0 and 1

👉 The propositions are usually of the type "Component X is in a failed state"

Introduction to Combinatorics: Probabilities

A is true with probability P1, B with probability P2

Proposition A →	1-P1	P1
Proposition B ↓		
1-P2	(1-P1) * (1-P2)	P1 * (1-P2)
P2	(1-P1) * P2	P1 * P2

AND: $P(A \wedge B) = P1 * P2$

OR: $P(A \vee B) = P1 * (1-P2) + (1-P1) * P2 + P1 * P2$
 $= 1 - [(1-P1) * (1-P2)]$
 $= P1 + P2 - P1 * P2$

General Formulas for AND / OR with n Inputs, NOT, XOR



**Precondition for these formulas:
Stochastically independent
events!**

□ **AND-Gate:**
$$P_{out} = \prod_{i=1}^n P_i$$

□ **OR-Gate:**
$$P_{out} = 1 - \prod_{i=1}^n (1 - P_i)$$

□ **NOT-Gate:**
$$P_{out} = 1 - P_{in} \quad (\text{only one input})$$

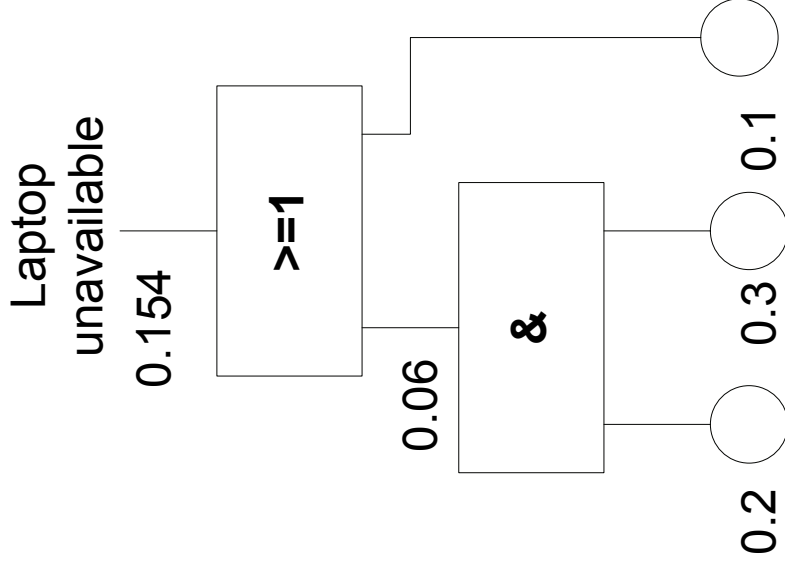
□ **XOR-Gate:**
$$P_{out} = \sum_{i=1}^n P_i \cdot \prod_{\substack{j=1 \\ j \neq i}}^n (1 - P_j)$$

 **XOR is normally defined for 2 inputs only**

- The **n-out-of-m Voter** can be replaced by a combination of AND / OR gates
- The **Inhibit-Gate** can be replaced by an AND and a NOT gate

 **The Priority-AND has no static combinatorial formula**

Calculation of Top-Event Probability



Battery empty Hardware
No socket defective
around

- ☐ Apply gate formulas in a bottom-up fashion
- ☐ Stop if top-event is reached

👉 **Bottom-up calculation is not efficient for large FTs**

There are two practical algorithms...

Calculation Method 1: Minimal Cut Sets

- The top-event is the union of all intersection-free minimal cut sets
- If cut-set probabilities are small (below 0.1), then intersection probabilities are even smaller
- The top-event probability is the sum of all MCS probabilities

$$P_{top} = \sum_{all\ MCS} P_{MCS\ i}$$

- The probability of each MCS is the product of the probabilities of the included basic events

$$P_{MCS} = \prod_{all\ events\ \in\ MCS} P_i$$

 **This algorithm leads to an approximation. It does not work for NOT gates**

Finding Minimal Cut Sets

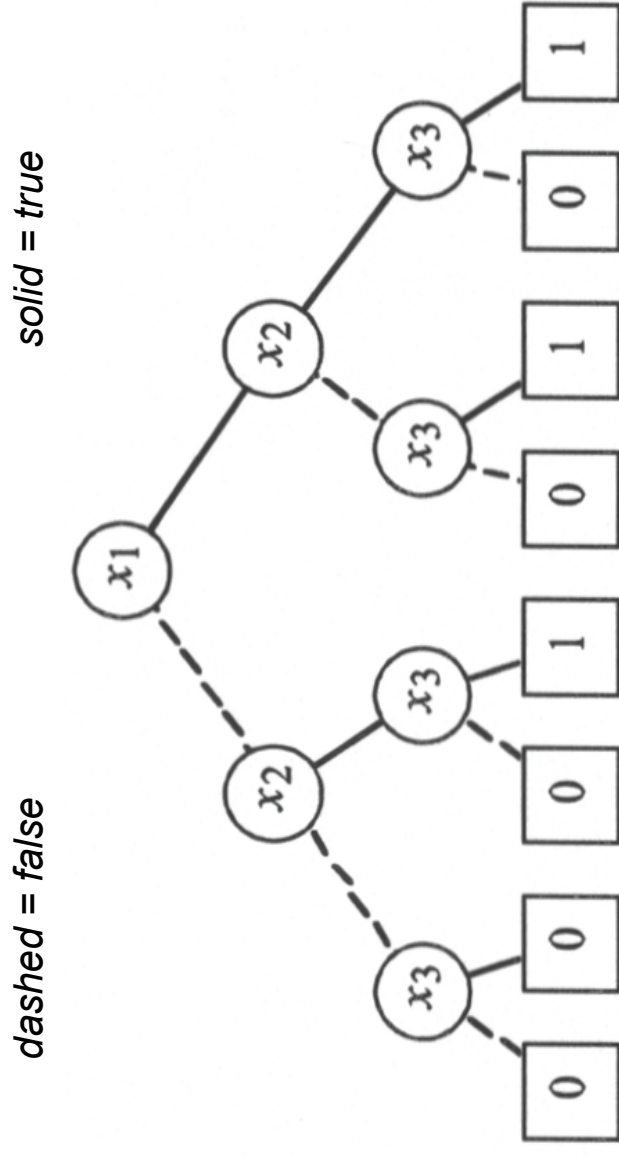
- ☐ Decompose the tree recursively
- ☐ For each OR gate
 - Generate as many entries as there are inputs $\{(i1), (i2), (i3)...\}$
- ☐ For each AND gate
 - Generate one entry containing all inputs $\{(i1, i2, i3,...)\}$
- ☐ Repeat until all gates are resolved
- ☐ Cancel cut sets that are not minimal (redundant)

Calculation Method 2: Using BDDs

- **BDD** = Binary Decision Diagram
- **OBDD** = Ordered BDD (defined variable order)
- **ROBDD** = Reduced Ordered BDD (after elimination of redundancies)
- ROBDDs are an efficient representation of Boolean formulas

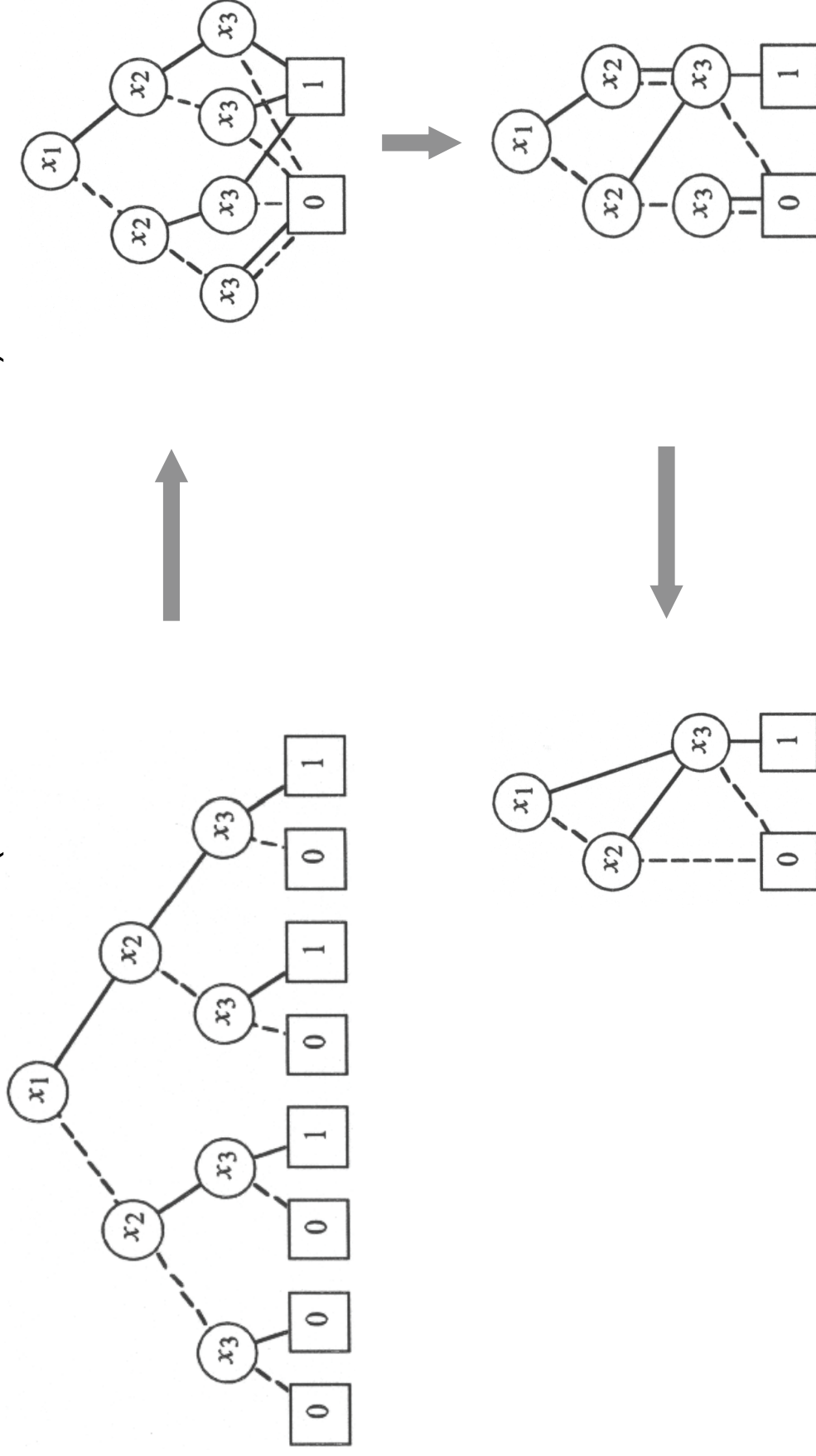
x_1	x_2	x_3	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

$$f = (x_1 \vee x_2) \wedge x_3$$



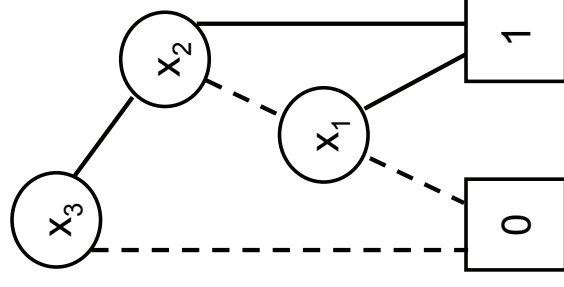
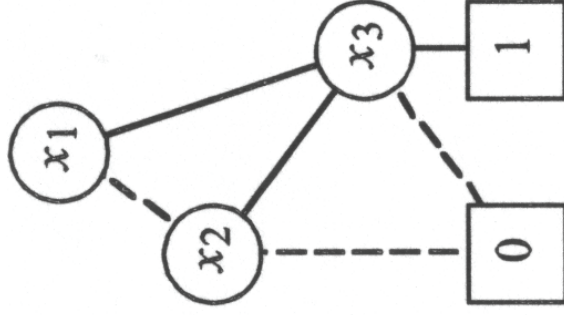
BDD Reduction

- Removal of redundant nodes (terminal and non-terminal) and tests:



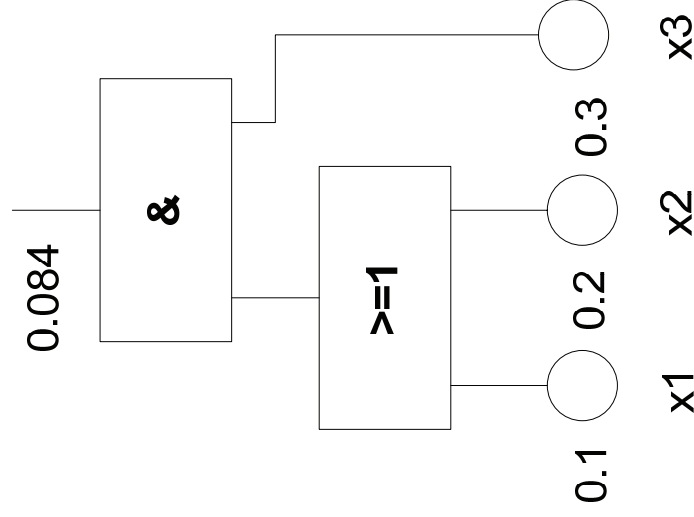
Impact of Variable Order

- The variable order may have considerable influence on the size of the OBDDs
- Same function, different variable order:

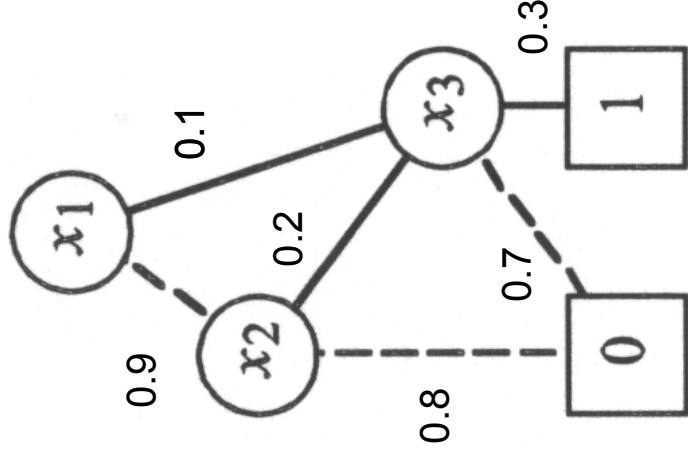


 **Finding the best variable order is NP-complete (= unachievable for large FTs)**

Calculation of Top-Event Probability via BDDs



- ☐ Annotate probabilities from FT to true branches
- ☐ Annotate 1-P to false branches
- ☐ For each path multiply branch probabilities
- ☐ Sum up all paths that lead to terminal 1



$$P = 0.1 * 0.3 + 0.9 * 0.2 * 0.3 = 0,084$$

Results beyond Top-Event Probability

- ☐ Probability of cut sets with order 1 (consisting of only one event)
 - Single points of failure should have extremely low probability
- ☐ Failure probabilities of (technical) sub-systems
 - Here, redundancy can reduce failure probability of the system
- ☐ Equivalent failure rates
 - Specify, which percentage of the intact systems are expected to fail within a given time span

Importance Measures

- ☐ Importance measures quantify the significance of FT events in terms of their contribution to the top-event probability
- ☐ To know the importance of part of the FT is important for
 - **Robustness Estimation:** How much will my result change if input values are roughly estimated or change during operation?
 - **Work planning:** You should rather spend your time on system changes that have significant impact on overall failure probability

Some Importance Measures

- ☐ Fussell-Vesely Importance
 - Absolute or relative (= percentage) contribution to the top-event probability
- ☐ Risk Reduction Worth or Top Decrease Sensitivity
 - Decrease of top-event probability if a given event is assured not to occur
- ☐ Risk Achievement Worth
 - Increase of top-event probability if a given event occurs
- ☐ Binbaum's Importance Measure
 - Rate of change of top-event probability in relation to rate of change of a given event

Other Issues in Quantitative Analysis

- ☐ Uncertainty Quantification
 - Event data is taken from samples or from other environment
 - Sensitivity analysis or formal uncertainty analysis (assigning a probability distribution)
- ☐ Coverage Factors
 - Take into account that some failures do not lead to catastrophic results
- ☐ Time or Phase Dependent Analysis
 - Use different models or rates for different time intervals according to mission phases

Literature

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□ BDD Algorithm

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