Safety and Reliability of Embedded Systems

(Sicherheit und Zuverlässigkeit eingebetteter Systeme)

Fault Tree Analysis
Mathematical Background and Algorithms

Content

- Definitions of Terms
- Introduction to Combinatorics
- General Formulas for AND-, OR-, NOT-, XOR-Gates
- Calculation of Top-Event Probability
- Results beyond Top-Event Probability
- Importance Measures
- Other Issues in Quantitative Analysis

Definitions of Terms

- **Failure** is any behavior of a component or system that deviates from the specification
- **Fault** is an abnormal state or condition within a component that can lead to a failure
- **Accident** is an undesired event that causes death or injury of persons or harm to goods or to the environment
- **Hazard** is a state of a system *and* its environment where the occurrence of an accident depends only on influences that are not controllable by the system
- **Risk** is the combination of hazard probability and severity of the resulting accident
- **Acceptable Risk** is a level of risk that authorities or other bodies have defined as acceptable according to acceptance criteria

Definitions of Terms

Other definitions exist, but many of them are unpractical

- **Safety** is freedom from unacceptable risks
  - **Safety analysis aims at proving that the actual risk is below the acceptable risk**
- **Availability** is the property of a system to fulfill its purpose at a given point in time / is the probability that the system fulfills its purpose at a given point in time
  - **The focus is on uninterrupted service**
- **Reliability** is the property of an entity to fulfill its reliability requirements during or after a given time span under given application conditions
  - **Reliability is related to the probability of a failure event over the mission time**

Modeling of Reliability

- Reliability Function R(t):
  - F(t) gives the probability that at time t the (non-repairable) system has failed
  - Thus R(t) = 1 - F(t) is the probability that at time t no failure has occurred yet
- Probability Density f(t):
  - The probability density f(t) describes the modification of the probability that a system fails over time:

$$f(t) = \frac{d\,F(t)}{dt}$$

States have a probability.
Events have a probability density and an (occurrence) rate

- Failure Rate:
  - The failure rate is the relative boundary value of failed entities at time t in a time interval that approximates zero, referring to the entities still functional at the beginning of the time interval:

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{dF(t)\,/\,dt}{R(t)} = \frac{-\,dR(t)\,/\,dt}{R(t)}$$

Cut Sets, Minimal Cut Sets, Path Sets

- A **Cut Set** is a set of basic events, which in conjunction cause the top event
- A **Minimal Cut Set (MCS)** is a cut set that no longer is a cut set if any of its basic events is removed
- A **Path Set** is a set of basic events that, if they are false, inhibit the top event from occurring
- A **Minimal Path Set (MPS)** is a path set that no longer is a path set if any of its basic events is removed

Introduction to Combinatorics: Truth Values

AND: $A \wedge B$ OR: $A \vee B$

Proposition **A** → Proposition **B** ↓	False	True
False	False *False*	False *True*
True	False *True*	True *True*

- True and False are often represented by 0 and 1
- The propositions are usually of the type "Component X is in a failed state"

Introduction to Combinatorics: Probabilities

A is true with probability P1, B with probability P2

Proposition **A** → Proposition **B** ↓	1-P1	P1
1-P2	(1-P1) * (1-P2)	P1 * (1-P2)
P2	(1–P1) * P2	P1 * P2

AND: $P(A \wedge B) = P1 * P2$

OR: $P(A \vee B) = P1 * (1-P2) + (1-P1) * P2 + P1 * P2$

$= 1 - [(1-P1)*(1-P2)]$

$= P1 + P2 - P1*P2$

General Formulas for AND / OR with n Inputs, NOT, XOR

- **AND-Gate**: $P_{out} = \prod_{i=1}^{n} P_i$
- **OR-Gate**: $P_{out} = 1 - \prod_{i=1}^{n} (1 - P_i)$
- **NOT-Gate**: $P_{out} = 1 - P_{in}$ **(only one input)**
- **XOR-Gate**: $P_{out} = \sum_{i=1}^{n} P_i \cdot \prod_{\substack{j=1 \\ j \neq i}}^{n} (1 - P_j)$

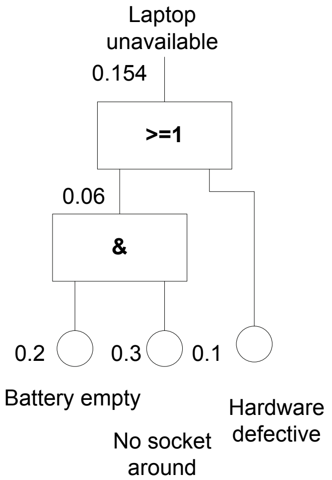
☝ **XOR is normally defined for 2 inputs only**

Precondition for these formulas: Stochastically independent events!

- The **n-out-of-m Voter** can be replaced by a combination of AND / OR gates
- The **Inhibit-Gate** can be replaced by an AND and a NOT gate

☝ **The Priority-AND has no static combinatorial formula**

Calculation of Top-Event Probability



- Apply gate formulas in a bottom-up fashion
- Stop if top-event is reached

☝ **Bottom-up calculation is not efficient for large FTs**

There are two practical algorithms...

Calculation Method 1: Minimal Cut Sets

- The top-event is the union of all intersection-free minimal cut sets
- If cut-set probabilities are small (below 0.1), then intersection probabilities are even smaller
- The top-event probability is the sum of all MCS probabilities

$$P_{top} = \sum_{all\ MCS} P_{MCS\ i}$$

- The probability of each MCS is the product of the probabilities of the included basic events

$$P_{MCS} = \prod_{all\ events \in MCS} P_i$$

☝ **This algorithm leads to an approximation. It does not work for NOT gates**

Finding Minimal Cut Sets

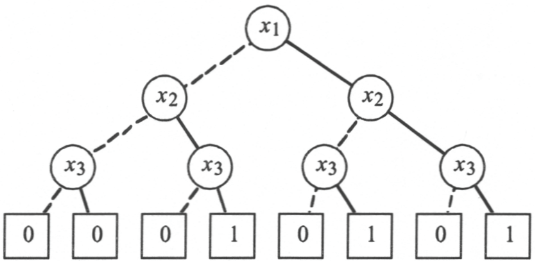
- Decompose the tree recursively
- For each OR gate
  - Generate as many entries as there are inputs {(i1), (i2), (i3)...}
- For each AND gate
  - Generate one entry containing all inputs {(i1, i2, i3,...)}
- Repeat until all gates are resolved
- Cancel cut sets that are not minimal (redundant)

Calculation Method 2: Using BDDs

- **BDD** = Binary Decision Diagram
- **OBDD** = Ordered BDD (defined variable order)
- **ROBDD** = Reduced Ordered BDD (after elimination of redundancies)
- ROBDDs are an efficient representation of Boolean formulas

dashed = false *solid = true*

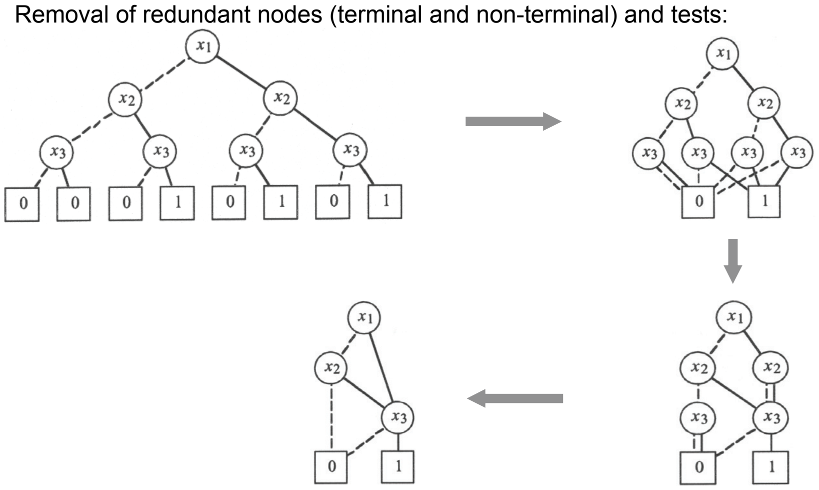
x_1	x_2	x_3	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1



f = (x1 ∨ x2) ∧ x3

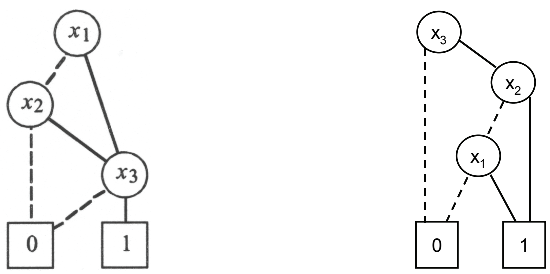
BDD Reduction

- Removal of redundant nodes (terminal and non-terminal) and tests:



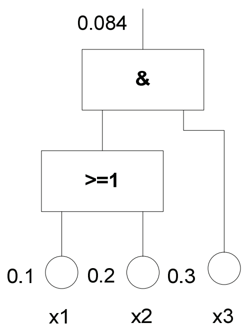
Impact of Variable Order

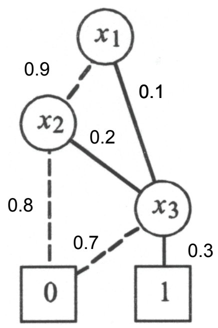
- The variable order may have considerable influence on the size of the OBDDs
- Same function, different variable order:



☝ **Finding the best variable order is NP-complete (= unachievable for large FTs)**

Calculation of Top-Event Probability via BDDs





- Annotate probabilities from FT to true branches
- Annotate 1-P to false branches
- For each path multiply branch probabilities
- Sum up all paths that lead to terminal 1

P = 0.1 * 0.3 + 0.9 * 0.2 * 0.3 = 0,084

Results beyond Top-Event Probability

- Probability of cut sets with order 1 (consisting of only one event)
  - Single points of failure should have extremely low probability
- Failure probabilities of (technical) sub-systems
  - Here, redundancy can reduce failure probability of the system
- Equivalent failure rates
  - Specify, which percentage of the intact systems are expected to fail within a given time span

Importance Measures

- Importance measures quantify the significance of FT events in terms of their contribution to the top-event probability
- To know the importance of part of the FT is important for
  - **Robustness Estimation**: How much will my result change if input values are roughly estimated or change during operation?
  - **Work planning**: You should rather spend your time on system changes that have significant impact on overall failure probability

Some Importance Measures

- Fussell-Vesely Importance
  - Absolute or relative (= percentage) contribution to the top-event probability
- Risk Reduction Worth or Top Decrease Sensitivity
  - Decrease of top-event probability if a given event is assured not to occur
- Risk Achievement Worth
  - Increase of top-event probability if a given event occurs
- Binbaum's Importance Measure
  - Rate of change of top-event probability in relation to rate of change of a given event

Other Issues in Quantitative Analysis

- Uncertainty Quantification
  - Event data is taken from samples or from other environment
  - Sensitivity analysis or formal uncertainty analysis (assigning a probability distribution)
- Coverage Factors
  - Take into account that some failures do not lead to catastrophic results
- Time or Phase Dependent Analysis
  - Use different models or rates for different time intervals according to mission phases

Literature

- FTA and Probabilistic Risk Assessment
  - Birolini A., Reliability engineering: theory and practice (third ed.), New York, Springer, 1999
  - Kececioglu, D.: Reliability Engineering Handbook, Vol. 1 & 2, PTR Prentice Hall, 1991.
- Minimal Cut Sets
  - Fussel JB, Vesely WE, A new methodology for obtaining cut sets for fault trees, in Trans. American Nuclear Society, 15, 262, 1972
  - T. Kohda & EJ Henley, On Digraphs, Fault Trees,and Cut Sets, Reliability Engineering and System Safety, 20 (1), 1988, p. 35-61
- BDD Algorithm
  - Bryant, R.E.: Graph-based algorithms for boolean function manipulation. IEEE Transactions on Computers, C-35(8):677--691, Aug. 1986.
  - O. Coudert and J. C. Madre. Fault tree analysis: 10^20 prime implicants and beyond. In Proceedings of the Annual Reliability and Maintainability Symposium, pages 240-5, Atlanta, GA, 26-28 January 1993
  - A. Rauzy. An Brief Introduction to Binary Decision Diagrams. RAIRO-APII-JESA, 30(8):1033-1051, 1996